Fermi National Accelerator Laboratory

FERMILAB-Pub-84/37-AST
March, 1984

More Dimensions - Less Entropy

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ABSTRACT

For a cosmological model with d non-compact and D compact spatial dimensions and symmetry $R^1 \times S^d \times S^D$, we calculate the entropy produced in d dimensions due to the compactification of D dimensions and show it too small to be of cosmological interest. Although insufficient entropy is produced in the model we study, the contraction of extra dimensions does lead to entropy production. We discuss modifications of our assumptions which may lead to a large entropy production.

I INTRODUCTION

The origin of the observed internal gauge symmetries from symmetries of an internal compact space is an attractive approach for the unification of particle physics with gravity.[1,2] The basic idea may be implemented by several approaches, a common assumption being that there are more than four space-time dimensions, with the extra dimensions unobservable today because they are compactified to a very small scale. The natural scale of the compactified dimensions is expected to be the Planck length.

As the energies necessary to probe the extra dimensions are a factor of 3×10^{14} larger than that of the largest proposed terrestrial accelerator, it is natural to attempt to use the primordial accelerator, the big-bang, to study the effect of extra dimensions. Several authors have suggested that the existence of extra dimensions might be responsible for the large observed entropy of the universe.[3-6] The basic reason for the entropy increase is that one may have epochs of increasing non-compact dimensions with decreasing mean volume due to the contraction of compactified dimensions. In an isentropic universe, the decrease in mean volume leads to an increase in temperature, even with increasing non-compact dimensions. The fact that the temperature increases during expansion of the non-compact dimensions may be interpreted as an increase in the effective entropy of the non-compact dimensions.[7]

In this paper we analyze entropy production in detail for a class of cosmological models in more than 4 dimensions and conclude that there is negligible entropy production. This is a disappointment, since a large entropy production from extra dimensions could, in principle, solve many cosmological problems, and might be considered a competitor of the usual inflationary models.[8]

We will consider cosmologies with N spatial dimensions and 1 time dimension. We will also assume the N spatial dimensions are split into d non-compact dimensions (large today) and D compact dimensions (small today). The particular cosmology we will study is described by a metric in the block diagonal form.

$$g_{MP}(x,y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & R_d^2(t)g_{mp}(x) & 0 \\ 0 & 0 & R_D^2(t)g_{\mu\tau}(y) \end{pmatrix} \tag{1.1}$$

Here $M,P = 0,1,\ldots.N$; $m,p, = 1,\ldots.d$; $\mu,\tau = d+1,\ldots.N$; $g_{mp}(x)$ and $g_{\mu\tau}(y)$ are the metrics for maximally symmetric d-dimensional and D-dimensional spaces; and $R_d(t)$ and $R_D(t)$ are the cosmological scale factors for the d-dimensional and D-dimensional spaces; x^i and y^ν are the coordinates for the d-dimensional and D-dimensional spaces.

Cosmologies with the metric (1.1) were first considered by Freund[9] in the context of Jordan-Brans-Dicke Theories, 11-dimensional supergravity, and 10-dimensional N = 1 supergravity. They were later considered in Kaluza-Klein theories by Sahdev.[6] We will follow the general approach of Sahdev, and assume the metric (1.1), the Einstein equations in N+1 dimensions without external fields or a cosmological constant, and that when the universe is "small enough" such that the curvature of the compact dimensions is negligible the N+1 dimensional stress energy tensor has the perfect fluid form. $(M,P = 0,1\ldots.N)$

$$T_{MP} = pg_{MP} + (p + \rho)U_M U_P \tag{1.2}$$

where p is pressure, ρ is the energy density, and U_M is the velocity N -vector of the fluid.

The solutions to the Einstein equations depend upon the curvature of the d-dimensional and the D-dimensional spaces. We will assume that the curvature of the d-dimensional space is zero, and that the curvature of the D-dimensional space is positive. As noted by Dicke and Peebles,[10] and emphasized by Guth,[11] the choice of a zero-curvature space is unnatural (AKA the flatness problem). A "natural" cosmology would have the curvature of both spaces comparable to some fundamental microphysics scale, say m_{pl} (where m_{pl} is the Planck mass), and the curvature of the d-dimensional space large today because of entropy creation (AKA inflation) at an earlier epoch. It is also possible that as entropy creation solves the flatness problem, it also solves the horizon problem by creating the observed entropy of the universe in a causal (hence possibly smooth) region of space. We will show in this paper that the entropy creation in the models we consider is insufficient for the above desiderata.

II THE FIELD EQUATIONS

With a metric given by (1.1) and a perfect fluid form for the stress-energy tensor (1.2) the Einstein equations, $R_{MP} - 1/2 g_{MP} R = -8\pi\bar{G}T_{MP}$, give

$$d\frac{\ddot{R}_d}{R_d} + D\frac{\ddot{R}_D}{R_D} = \frac{8\pi\bar{G}}{N-1}\left[\rho(2-N)-pN\right] \tag{2.1.a}$$

$$\frac{d}{dt}\left(\frac{\dot{R}_d}{R_d}\right) + \frac{\dot{R}_d}{R_d}\left(d\frac{\dot{R}_d}{R_d} + D\frac{\dot{R}_D}{R_D}\right) = \frac{8\pi\bar{G}}{N-1}(\rho - p) \tag{2.1.b}$$

$$R_D^{-2} + \frac{d}{dt}\left(\frac{\dot{R}_D}{R_D}\right) + \frac{\dot{R}_D}{R_D}\left(d\frac{\dot{R}_d}{R_d} + D\frac{\dot{R}_D}{R_D}\right) = \frac{8\pi\bar{G}}{N-1}(\rho - p), \tag{2.1.c}$$

where N is the number of spatial dimensions, $N = D + d$. In (2.1), $\bar{G}$ is the fundamental constant that appears in the gravitational part of the $N + 1$ dimensional action ($\bar{G}$ has dimension of length $^{N-1}$)

$$I_{N+1} = -(16\pi\bar{G})^{-1} \int d^{N+1}x \quad g_{N+1} \quad R_{N+1} \tag{2.2}$$

where g_{N+1} and R_{N+1} are the $(N+1)$-dimensional metric determinant and scalar curvature. Upon integration of (2.2) over the extra dimensions to obtain an effective 4-dimensional gravitational action, $\bar{G}$ is related to Newton's constant, G_N, by

$$\bar{G} = G_N V_D, \tag{2.3}$$

where V_D is the volume of the compact D-dimensional space.

In the Kaluza-Klein approach all fields are massless in higher dimensions. Therefore we will assume an equation of state for radiation, $Np = \rho$. The conservation law $T^{MP}{}_{;P} = 0$ gives

$$\rho\sigma^{N+1} = \rho_o\sigma_o{}^{N+1} = \text{constant}, \tag{2.4}$$

where σ^N is the dimensionless mean spatial volume given by (α is some arbitrary constant with dimension length^{-1})

$$\sigma = \alpha\ (R_d^d\ R_D^D)^{1/N}, \tag{2.5}$$

and ρ_o and σ_o are the appropriate quanities defined at an arbitrary e

In principle it is straightforward to solve the field equations for $R_D(t)$ and $R_d(t)$. However in practice it is convenient to make some changes of variables. Let us change the time coordinate according to (γ is an arbitrary constant with dimension of length)

$$dt = \gamma\sigma(x)dx. \tag{2.6}$$

Since $\sigma(x)$ is a positive function of time, the new co-ordinate x is monotonic in t.

In terms of σ and x, (2.1.b) has the first integral

$$\frac{R_d'}{R_d} = \frac{B\ (x - x_o)}{\sigma^{N-1}} \tag{2.7}$$

where ' denotes d/dx, x_o is an integration constant, and B is a dimensionless constant given by ($\bar{G}$ has dimension length $^{N-1}$; σ_o is dimensionless; ρ_o has dimension length^{-N-1})

$$B = \frac{8\pi}{N}\ \bar{G}\ \gamma^2 \rho_o\, \sigma_o^{N+1} \tag{2.8}$$

For simplicity we choose the magnitude of γ to give B = 1. With the substitutions (2.5) and (2.6), (2.1.c) becomes

$$\frac{(N-1) - (\sigma^{N-1})''}{\sigma^{N-1}} = \frac{(N-1)\ D}{N}\ \gamma^2\ \alpha^{2N/D}\ \left(\frac{R_d}{\sigma}\right)^{2d/D} \tag{2.9}$$

and (2.1.a) becomes

$$\sigma^{N-1}\ (\sigma^{N-1})'' + \left[\ \frac{d}{D(N-1)} - 1\right]\ (\sigma^{N-1})'^2 - 2\frac{d}{D}\ (x-x_o)(\sigma^{N-1})'$$

$$+ (N-1)\sigma^{N-1} + (N-1)\ \frac{d}{D}\ (x-x_o)^2 = 0. \tag{2.10}$$

Note that x appears in these equations only in derivatives or in the combination $(x-x_o)$; consequently we can translate x without altering the form of the equations, and in particular we can always choose to set $x_o = 0$ as different values of x_o cannot lead to physically different solutions.

We choose the remaining arbitrary constant α to satisfy $\alpha^{N/D}\gamma = N$, and make two further changes of variable

$$\omega = \sigma^{N-1} \quad ; \quad v = \sigma^{2N(D-1)/D}\ R_d^{\ 2d/D} \tag{2.11}$$

to render the system into the final form

$$\omega\omega'' + \left(\frac{d}{D(N-1)} - 1\right)\omega'^2 - \frac{2d}{D}\, x\omega' + (N-1)\omega + (N-1)\,\frac{d}{D}\, x^2 = 0 \qquad (2.12.a)$$

$$(N-1)\omega - \omega\omega'' = N(N-1)\; Dv \qquad (2.12.b)$$

$$\frac{D}{2d}\;\frac{v'}{v} - \frac{N(D-1)}{d(N-1)}\;\frac{\omega'}{\omega} = \frac{x}{\omega} \qquad (2.12.c)$$

The first of these is a second order equation in ω alone, and it is not difficult to verify that any solution of this equation, when used to determine v through the second equation, also satisfies the third. The cause of this apparent redundancy is that we have performed an integration in going from (2.1) to (2.12), and have thereby reduced the system to only two independent equations.

It is worthwhile to have an expression for R_D and R_d in terms of ω and v

$$R_D = \alpha^{-N/D}\,\omega^{N/(N-1)}\,v^{-1/2} \qquad (2.13.a)$$

$$R_d = \omega^{N(1-D)/d(N-1)}\,v^{D/2d} \qquad (2.13.b)$$

In the next section we give the solutions to the system of equations (2.12).

III SOLUTIONS TO THE EQUATIONS OF MOTION

In this section we describe solutions to the equations of motion. In the special case d = 1 and D = 2 there is a simple analytic solution first found by Sahdev.[6] This analytic solution is discussed in Appendix A. For more interesting values of d and D it is necessary to solve the equations numerically. We will integrate the equations assuming that at $x = 0$, the mean volume vanishes, i.e. $\omega(x=0) = 0$. By performing a power series expansion around $x = 0$, we find

$$\omega = \frac{N-1}{2} x^2 - \frac{ND}{6} a x^4 + O(x^6) \tag{3.1.a}$$

$$v = a x^4 + O(x^6) \tag{3.1.b}$$

where a is an arbitrary positive constant. By doing a similar expansion one finds that it is impossible to make both R_D and R_d vanish about a non-zero value of x. Therefore the only solutions which start out with both the compact and the open scale-factor zero must begin at $x = 0$. (This does not contradict the time-translation invariance of the original equations, since the relation of t to x includes an arbitrary constant of integration.) Using the definitions of R_D and R_d in terms of v and ω, one finds simple power-law behavior near the initial singularity;

$$R_D = \alpha^{-N/D}\, a^{-1/2} \left(\frac{N-1}{2}\right)^{N/(N-1)} x^{2/(N-1)} \tag{3.2.a}$$

$$R_d = \left(\frac{N-1}{2}\right)^{N(1-D)/d(N-1)} a^{D/2d}\, x^{2/(N-1)} \tag{3.2.b}$$

It is not surprising that the two scale-factors start off with similar behavior. It is a familiar result from conventional Robertson-Walker models that curvature terms are negligible near the initial singularity.

Once we know the behavior of the solutions near x = 0, we can use the equations of motion to integrate the system. In Figure 1 we give an example of a solution for d = 3 open and D = 7 compact dimensions. The behavior of R_D and R_d is qualitatively the same for any values of d and D. In all solutions one finds that R_D starts from zero at an initial singularity, increases to a maximum, and then vanishes at a second singularity. R_d also starts from zero, with the same growth rate as R_D. However, at the final singularity R_d always becomes infinite. Another general characteristic of the solutions is that soon after the compact dimensions reach R_M, the mean volume of the scale factors actually decreases even though the scale factor for the usual dimensions continues to increase. The decrease in the mean volume leads to an increase in the temperature so long as the expansion is isentropic. This increase in temperature while the d-dimensional space expands leads to an increase in entropy density. In Figure 2 we plot the temperature as a function of x for the model of Figure 1. We note that as the second singularity is approached, the temperature (hence the entropy density) diverges. However, as we discuss in the next section, we cannot use our equations arbitrarily close to the singularity.

We have a single parameter set of solutions for ω and v, and in addition there are two relations between the three constants α, Υ and B. Apparently we then have a two-parameter family of solutions, but a gauge degree of freedom removes one of them. This happens since $k_d = 0$, and we can change R_d by a constant scaling without producing physically different solutions. The other scale-factor R_D does not have this scaling degree of freedom because we have made the choice $k_D = 1$.

The existence of a single parameter dependence in the solutions can be made clear by exhibiting the scaling behavior of the solutions for different constants a in (3.1). If ω and v are a pair of functions satisfying the field equations, it is easily seen that $b^2\omega(x/b)$ and $b^2 v(x/b)$ are also solutions. Two quantities of physical significance in the solution are the compact scale-factor R_D and the density ρ. These scale with ω and v according to

$$R_D = \alpha^{-N/D}\, b^{(N+1)/(N-1)}\, \omega^{N/(N-1)}\, v^{-1/2} \tag{3.3}$$

$$\rho = \rho_o \sigma_o^{N+1}\, \sigma^{-(N+1)} = \frac{\alpha^{2N/D}}{8\pi NG}\, b^{-2(N+1)/(N-1)}\, \omega^{-(N+1)/(N-1)} \tag{3.4}$$

where to get (3.4) we have used the fact that we have set B = 1. Because of the gauge degree of freedom, neither b nor α has an unambiguous physical interpretation, but we can see from the above that the combination $\alpha^{-N/D}\, b^{(N+1)/(N-1)}$ appears in the definition of R_D and ρ, and can therefore be taken as a physically significant parameter in the solutions.

Let us suppose that we have found a pair of functions ω and v such that the combination $\omega^{N/(N-1)}\ v^{-1/2}$ has a maximum value of one. Using the scaling derived above, this allows us to parametrize our set of solutions as

$$R_D = R_M \ \omega^{N/(N-1)}\ v^{-1/2} \tag{3.5}$$

$$\rho = (8\pi N\bar{G}R_M^2)^{-1}\ \omega^{-(N+1)/(N-1)} \tag{3.6}$$

The quantity R_M is simply the maximum value of R, and since the curvature of the compact dimensions has been normalized to one, it is the maximum physical size attained by the extra dimensions. This is a convenient way to parametrize the complete set of solutions.

IV DECOUPLING OF THE EXTRA DIMENSIONS

Since the extra dimensions are presumably stable today, we must assume that before the second singularity a miracle occurs to stabilize the extra dimensions. A miracle is necessary for stabilization of the extra dimensions since for $D > 1$, $M^4 \times B^D$ (B^D is a compact D-dimensional space) is not a static vacuum solution to the Einstein equations. The miracle may be in the form of additional matter fields,[12] or a cosmological constant.[13] In this paper we do not attempt to explain the miracle, but rather push the equations as far as we can before we know they fail.

A critical assumption in deriving the dynamical equations of this model Universe is the perfect-fluid form of the stress-energy tensor. We might expect this assumption to fail first when the difference in expansion rate between the open and compact dimension becomes large, because perfect fluid behavior requires local isotropy in the distributions of particles, and this in turn demands that energy can be redistributed along different dimensions faster than the relative expansion rate, i.e. $|\dot{R}_d/R_d - \dot{R}_D/R_D| > \Gamma$, where Γ is a typical reaction rate. Since before freeze-out all the particles are massless, on dimensional grounds we expect $\Gamma \sim T$. Therefore a rough criterion for perfect fluid behavior might be $\dot{R}/RT < 1$, where R refers to either R_d or R_D. The same problem arises in conventional cosmological models with anisotropy, and the failure of the perfect fluid approximation leads to an effective equation of state with non-zero viscosity[14]. However, the qualitative behavior of solutions does not change; expanding or contracting dimensions continue to expand or contract, although at different rates.

Proper consideration of these problems is beyond the scope of this paper. Instead, we shall assume perfect fluid behavior throughout the model, until a critical point is reached when the wavelength of excitations in the compact dimensions ($\lambda \sim T^{-1}$) becomes equal to the radius of the compact space. At this point, perfect fluid behavior fails for a more serious reason, in that the concept of a classical stress-energy tensor breaks down; the permissible energy states in the compact dimensions are quantized by the finite size. The transition from classical behavior occurs roughly when $R_D T = 1$, T being the temperature calculated from $\rho = T^{N+1}$. At this point the excitations of the extra dimensions correspond to massive particles, $m \cong R_D^{-1} > T$, and the pressure in the extra dimensions is effectively zero. The excitations in the non-compact dimensions describe massless particles with $p = d\rho$, so the assumption of isotropy for the stress-energy tensor fails and we cannot use our equations beyond this point.

What happens next is a difficult question. At some stage we want the compact dimensions to cease contracting and become stabilized at a finite radius. This may be achieved by a suitable change in the stress-energy tensor. Since every Kaluza-Klein model is entitled to at least one miracle, the simplest assumption is to invoke ours here and assume that freeze-out and stabilization occurs at the same point[15]. If we do this then our model is completely calculable up to freeze-out, and immediately afterwards it is assumed that the open dimensions continue to expand as a conventional Robertson-Walker Universe. A possible alternative is that after freeze-out the compact dimensions continue to contract (and are still coupled to the open dimensions), with a semiclassical stress-energy tensor whose form we can only guess at, and at some time later stabilization occurs. The difficulty with this is that it needs two miracles instead of one, and it leaves a grey area between freeze-out and stabilization in which the field equations are unknown. For the remainder of this section we will assume freeze-out and stabilization are simultaneous. In section VI we will generalize this restriction and show that entropy production is a maximum if freeze-out and stabilization are concurrent.

From (3.5) and (3.6) we have

$$\rho R_D^{N+1} = \frac{R_M^{N-1}}{8\pi \bar{G} N} (\omega v^{-1/2})^{N+1} \tag{4.1}$$

Freeze-out occurs when $T = R_D^{-1}$, and since $\rho = T^{N+1}$, the left hand side of (4.1) is one at freeze-out. We denote by a subscripted * the values of the parameters at freeze-out. From (4.1)

$$(\omega_* v_*^{-1/2})^{N+1} = 8\pi N \bar{G}\, R_M^{1-N} \tag{4.2}$$

Since we are assuming R_* is the present distance scale of the extra dimensions, R_* is related to $\bar{G}$ by

$$\bar{G} = V_D\, R_{pl}^{d-1} = R_*^D\, R_{pl}^{d-1} \tag{4.3}$$

where R_{pl} is the Planck length and in (4.3) we have ignored factors which appear in the formula for $V_D = 2\pi^{(D+1)/2}/\Gamma[(D+1)/2]$.

We can now express three dimensionless length ratios in terms of v_* and ω_*

$$(R_*/R_M) = \omega_*^{N/(N-1)} v_*^{-1/2} \tag{4.4.a}$$

$$(R_M/R_{pl})^{d-1} = 8\pi N v_*^{(d+1)/2} \omega_*^{(1-Nd)/(N-1)} \tag{4.4.b}$$

$$(R_*/R_{pl})^{d-1} = 8\pi N v_*/\omega_* \ . \tag{4.4.c}$$

As discussed in the previous section, for particular values of D and d there is a one-parameter family of solutions generated by, for instance, different values of the constant a in the initial values of ω and v [see (3.1)]. For a given value of a one may integrate the equations forward in time until freeze-out at $T = R_*^{-1}$. Different values of a would correspond to different values of (R_*/R_{pl}) if we stop at the same value of x_*. However, one may take advantage of the invariance of the equations under scale transformations $\{\omega(x)\to b^2\omega(x/b); v(x)\to b^2 v(x/b)\}$ to study the entire family of solutions from the numerical solution for a single value of a, stopping at different values of x_*. For this single value of a, the dynamical equations are integrated numerically to produce functions $\omega(x)$ and $v(x)$; because of the scaling relation between all solutions of the equations, we can stop at any point on the calculated solution and demand that freeze-out should occur there. Equations (4.4) then give the ratios of the three physical lengths in the problem, R_M, R_*, and R_{pl}. Demanding that freeze-out occur at different points in the integration (giving any desired length ratios) is equivalent to choosing different values of a to give the desired length ratios at freeze-out. How this is accomplished is illustrated in Appendix A for the D = 2, d = 1 analytic solutions.

In the limit as the freeze-out epoch gets close to the final singularity, we can approximate ω and v by leading terms in a power-law expansion. Putting $y = x_s - x$, where x_s is the position of the final singularity, we obtain

$$\omega = x_s \frac{d(N-1)}{N(D-1)} [1 + q]\, y + O(y)^2 \tag{4.5.a}$$

$$v = b_1\, y^{2[1 - 1/q]} \tag{4.5.b}$$

where b_1 is a constant, and we have defined $q = [D(N-1)/d]^{1/2}$. (These power-laws are valid for D > 1 only; the special case D = 1 will be dealt with separately). Note that the constant in front of v is not determined except by numerically calculating a complete solution and matching it onto the approximation. The behavior of R_D and R_d near x_s

is

$$R_D \sim y^{1/(N-1) + 1/q} \tag{4.6.a}$$

$$R_d \sim y^{(1-q)/(N-1)} \tag{4.6.b}$$

which supports our previous assertion that R_D and R_d cannot both go to zero except at $x = 0$ (the power of y in R_d is ≤ 0 for $D > 1$). The relative scaling of the three lengths near the final singularity is

$$\left(\frac{R_M}{R_{pl}}\right)^{d-1} \sim y^{[(D/(N-1) - (d+1)/q]} \tag{4.7.a}$$

$$\left(\frac{R_*}{R_{pl}}\right)^{d-1} \sim y^{1-2/q} \tag{4.7.b}$$

$$\frac{R_*}{R_M} \sim y^{1/(N-1) + 1/q} \quad . \tag{4.7.c}$$

In the special case of only one extra dimension, the form of the functions ω and v near the final singularity is slightly different because the ω'^2 term in equation (2.12a) vanishes. The singularity becomes logarithmic, with ω and v going as

$$\omega \sim 2d\, x_s y \ln(1/y) \tag{4.8.a}$$

$$v \sim 4d\,(d+1)\, x_s^2 \ln(1/y). \tag{4.8.b}$$

For $D = 1$ the two scale factors near the singularity behave as

$$R_D \sim y^{(d+1)/d}\,[\ln(1/y)]^{(d+2)/2d} \tag{4.9.a}$$

$$R_d \sim [\ln(1/y)]^{1/2d} \tag{4.9.b}$$

and the length ratios behave as

$$R_*/R_M \sim \left(\frac{2x_s}{d+1}\right)^{1/2} y^{(d+1)/d}\,[\ln(1/y)]^{(d+2)/2d} \tag{4.10.a}$$

$$(R_*/R_{pl})^{d-1} \sim 16\pi x_s\, y^{-1} \tag{4.10.b}$$

$$(R_M/R_{pl})^{d-1} \sim \frac{(2dX_s)^{1/d}}{[d(d+1)]^{(d+1)/2}}\, y^{-(d^2+d-1)/d}\,[\ln(1/y)]^{-(d+2)(d-1)/2d} \tag{4.10.c}$$

V ENTROPY PRODUCTION

The quantity of physical interest is the entropy contained in a horizon volume after freeze-out. If the entropy is large, $S > 10^{88}$, then one is justified in starting a (3 + 1)-dimensional cosmological model at $T = T_*$ with initial conditions that are smooth over distances larger than the Hubble length (i.e. a Friedmann-Robertson-Walker model), and curvature small compared to the energy density. In this section we calculate the entropy in the horizon volume after freeze-out and show that it is small.

To calculate the entropy density in the non-compact dimensions we first take the N-dimensional energy density at stabilization, $\rho_N \sim T_*^{N+1}$ and integrate over the volume of the extra dimensions to calculate the d-dimensional energy density:

$$\rho_d = V_D T_*^{N+1} \tag{5.1}$$

We then assume that the energy density is instantly thermalized to give an entropy density $s_d = \rho_d^{d/(d+1)}$. Therefore the d-dimensional entropy density is given by

$$\begin{aligned} s_d &= [V_D \, T_*^{N+1}]^{d/(d+1)} \\ &= T_*^d = R_*^{-d} \end{aligned} \tag{5.2}$$

where we have used $V_D = R_*^D$, and $T_* R_* = 1$.

In the d-dimensional space, the horizon length is defined in the standard way as

$$l_h = R_d(t) \int_0^t dt' \, [R_d(t')]^{-1} = \gamma R_d(x) \int_0^x \frac{\sigma(x')}{R_d(x')} \, dx'. \tag{5.3}$$

Note that l_h is independent of the scale chosen for R_d, as of course it must be since l_h is a measurable quantity.

The entropy in the horizon volume is given by $S_d = s_d l_h^d = l_h^d / R_*^d$. In terms of ω and v

$$(S_d)^{1/D} = N\omega^{-N/d} \, v^{N/2d} \int_0^x \omega^{D/d} \, v^{-D/2d} \, dx' \tag{5.4}$$

where the factor N in front arises from the combination $\gamma\alpha^{N/D}$. This expression is easily checked to be invariant under the scaling discussed in Section III.

Using the expansions of the previous section for the behaviour of ω and v close to the final singularity, we find for D > 1

$$S_d \sim y^{-N/q} \left[\int_0^{x_s - y} \omega^{D/d} v^{-D/2d} dx'\right]^d . \qquad (5.5)$$

Here, as before, $y = x_s - x$, where x_s is the final singularity. The integrand in (5.5) goes to zero as the singularity is approached, so the leading behavior is the power-law in front of the integral. This means that the entropy in a horizon volume at freeze-out gets indefinitely large as freeze-out occurs closer to the singularity. From (4.7.c) it can be seen that the ratio of the freeze-out scale to the maximum size of the compact dimensions goes to zero as the singularity is approached, but the other two ratios may vanish or diverge depending on the number of dimensions.

The fact that the entropy in a horizon volume gets indefinitely large near the final singularity seems to suggest that extra dimensions can solve the entropy problem. However, in any acceptable model, we demand

$$\frac{R_*}{R_{pl}} \gtrsim O(1) , \qquad (5.6)$$

in other words, the Kaluza-Klein scale must be greater than the Planck scale.

In Figure 3 we give the result of numerical calculations for the entropy assuming $d = 3$. From the figure it is seen that if $R_*/R_{pl} = 10^{0\pm2}$, then neglible entropy is created. This is expected, since if we use (4.7.b) in (5.5) to calculate the leading behavior of S_3, we find (for $D > 1$)

$$\ln S_3 \sim \frac{-2(D+3)[3/D(D+2)]^{1/2}}{1-2[3/D(D+2)]^{1/2}} \ln (R_*/R_{pl})$$

$$\sim -2\sqrt{3}\ln(R_*/R_{pl}) \qquad (D >> 3). \tag{5.7}$$

Note that for $D \leq 2$, the entropy is maximal for $R_*/R_{pl} >> 1$, while for $D \geq 3$, the entropy is maximal for $R_*/R_{pl} << 1$. However S_3 is O(1) if R_*/R_{pl} is O(1).

Note that if we relax our assumption that $R_* = R_{KK}$ and allow a period of additional decrease in R_D before stabilization, there will be no additional entropy production.

VI CONCLUSIONS

The standard Friedmann-Robertson-Walker cosmology based upon the symmetry $R^1 \times S^3$ is a remarkably successful model. However it has several undesirable features. The effective curvature of S^3 is today much different than any reasonable microphysics scale (such as the Planck scale). Homogeneity and isotropy are initial conditions of the model and do not follow from any reasonable principle. Finally, the standard model has particle horizons - the universe we see now was causally disconnected at earlier times.

Guth[11] has shown that creation of a large amount of entropy at an early epoch can explain homogeneity and isotropy , and while it doesn't remove particle horizons, it can push the horizon "out-of-sight". In the inflationary universe picture the entropy production is the result of the release of a latent heat in some cosmological phase transition associated with spontaneous symmetry breaking. While this approach may indeed prove to be the origin of the entropy, it is worthwhile to investigate other possible origins of entropy production.

In this paper we have chosen a particularly simple extension of the standard model to accomodate extra dimensions; namely a cosmology based on the symmetry $R^1 \times S^d \times S^D$, where d is some number of dimensions large today, and D is some number of compact dimensions. In fact we have only studied the case where the curvature of S^d is zero (i.e. $S^d \rightarrow R^d$). However, if a large amount of entropy is produced in the freeze-out of the D-dimensions, so long as the curvature of S^d is somewhat smaller than the curvature of S^D, the effective curvature of S^d at early times would be zero.

A desirable cosmology might be one based upon $R^1 \times S^d \times S^D$, where the curvature of S^D is somewhat larger than that of S^d, but both are comparable to some microphysics scale. The scale factor associated with S^D will reach some maximum value and start to decrease. As it starts to decrease the mean volume will decrease, hence in an isentropic expansion the temperature will increase. Eventually the scale factor decreases sufficiently fast such that the curvature of the D-dimensions is smaller than the temperature and the modes associated with excitations of the extra dimensions will freeze-out, releasing entropy into d-dimensional excitations. The entropy release effectively inflates the d-dimensional space, giving all the desiderata of inflationary models.

By explicit numerical calculations we have shown that the above proposed cosmology cannot work. Since gauge coupling constants are related to ratios of the Kaluza-Klein scale to the Planck scale, the ratio cannot be much different from unity. The entropy created at freeze-out, however, scales as the ratio of the Kaluza-Klein scale to the Planck scale, and is much less than necessary to solve any cosmological problems. For instance for $D >> 3$, using (5.7), to have $S_3 \geq 10^{88}$ we would have to set $R_*/R_{pl} \sim 10^{-25}$, a quite unreasonable value. For $D = 3$, $S_3 \sim (R_*/R_{pl})^{-50}$, and sufficient entropy might be produced with $R_*/R_{pl} = 10^{-2}$. However, since gauge coupling constants are proportional to R_{pl}/R_*, we expect $R_{pl} < R_*$. In Figure 4 we give S_3 as a function of D for $R_*/R_{pl} = 1$. As D increases, S_3 decreases - hence the title of the paper.

Although the model we considered did not produce enough entropy, the basic idea of a temperature increase in expanding dimensions due to the contraction of compactified dimensions is an attractive, simple method for increasing the entropy density. The approach is sufficiently attractive that we now discuss some ways to relax the assumptions we made so as to allow enough entropy production. The first assumption was that the metric should be a solution of Einstein's equations. However, since we want the extra dimensions static today, we must add either external matter fields,[12] or a cosmological constant.[13] It may be possible that these modifications to the field equations may change the solutions enough to either steepen the dependence of S_3 upon R_*/R_{pl}, or shift the curves in Figure 3 to larger values of R_*/R_{pl}. It may also be the case that there are two extra dimensions which have nothing to do with the observed gauge symmetrics. For instance if $D = 2$ and $R_* = 3x10^3 R_{pl}$, then sufficient entropy is produced. Two compact dimensions are insufficient to give all the observed low energy gauge symmetrics, but the resulting gauge symmetrics from the two-dimensional compact space may have all their low-mass particles ($m \lesssim m_{pl}$) be gauge singlets, and hence unobservable today.[16] Then there is the possibility that stabilization occurs when $RT > 1$. In this case it is possible to create enough entropy and still have R_*/R_{pl} of $O(1)$. The entropy within a horizon volume is (for $D \gg 3$) $S_3 \sim (RT)^{5D/2}$. By choosing $RT = 10$, $D = 36$ it is possible to create enough entropy and not have any unnaturally large numbers built into the model. We thank Abbott, Barr and Ellis for bringing this last point to our attention.[16]

This work was supported in part by the Department of Energy and NASA.

Appendix A

In this appendix we use the analytic solutions to the field equations found by Sahdev[6] to demonstrate the scaling of the equations discussed in Section 3, and to illustrate by an analytic example the numerical calculation of the entropy[17].

The analytical solution is for the case D = 2, d = 1. The field equations (2.12) become

$$\omega\omega'' - \frac{3}{4}\omega'^2 - x\omega' + 2\omega + x^2 = 0 \tag{A.1.a}$$

$$2\omega - \omega\omega' = 12v \tag{A.1.b}$$

$$\frac{v'}{v} - \frac{3}{2}\frac{\omega'}{\omega} = \frac{x}{\omega} \quad , \tag{A.1.c}$$

which have solutions given by

$$\omega = x^2 - ax^4 \tag{A.2.a}$$

$$v = ax^2(x^2 - ax^4) \tag{A.2.b}$$

where a is an arbitrary (positive) constant that generates the expected one-parameter family of solutions. Note that for small x

$$\omega = \frac{N-1}{2}x^2 - \frac{ND}{6}ax^4 + O(x^6) \tag{A.3.a}$$

$$v = ax^4 + O(x^6) \tag{A.3.b}$$

as given in (3.1).

The compact and non-compact scale factors are given by (2.13)

$$R_D = \alpha^{-N/D}\,\omega^{N/N-1}\,v^{-1/2} = \alpha^{-3/2}(x-ax^3)^{-1/2} \tag{A.4.a}$$

$$R_d = \omega^{N(1-D)/d(N-1)}v^{D/2d} = ax^2(x^2-ax^4)^{-1/2}, \tag{A.4.b}$$

and the mean volume is

$$\sigma = \omega^{1/(N-1)} = (x^2-ax^4)^{1/2}. \tag{A.5}$$

Notice that at the second singularity, $x_s = 1/\sqrt{a}$, the mean volume vanishes ($\sigma \to o$), while the scale factor for the non-compact dimension diverges ($R_d \to \infty$).

As discussed in Section 3, if $\omega(x)$ and $v(x)$ are solutions, then

$$b^2\omega\ (x/b) = x^2 - \frac{a}{b^2}\, x^4 \tag{A.6.a}$$

$$b^2 v\ (x/b) = \frac{a}{b^2}\, x^2\ (x^2 - \frac{a}{b^2}\, x^4) \tag{A.6.b}$$

are also solutions, as can be easily seen as (A.6) is the same as (A.2) with $a \to a/b^2$. Thus if we have solutions to the field equations for one value of a, all other solutions may be generated by scaling $\{\omega(x) \to b^2\omega(x/b),\ v(x) \to b^2 v(x/b)\}$, as the scaling is equivalent to a solution with a different value of a.

R_D has a maximum value of $R_M = 2\alpha^{-3/2}/3a\sqrt{3}$ which occurs at $x_M = (3a)^{-1/2}$. Therefore we can express R_D as

$$R_D = R_M\, \frac{3\sqrt{3}}{2}\, a\ \ \frac{(x - ax^3)}{a^{1/2}} \tag{A.7}$$

and for a particular value of a ($a_1 = 2/3\sqrt{3}$) we will have $R_D = R_M\, \omega^{3/2}/v^{1/2}$, i.e. $\alpha^{-3/2} = R_M$. Let us call the functions ω and v with $a = a_1$, ω_1 and v_1. With $R_M = \alpha^{-3/2}$, the expression for the constant B becomes (after substituting $\gamma = N\alpha^{-N/D}$)

$$B = 8\pi N\bar{G}\ \alpha^{-2N/D}\ \rho_o\sigma_o^{N+1}$$

$$= 24\pi\bar{G}\ R_M^2\ \rho_o\sigma_o^4 \tag{A.8}$$

using $\rho_o\sigma_o^{N+1}$ = constant and B = 1,

$$\rho = \frac{1}{24\pi\bar{G}R_M^2\sigma^4} = \frac{1}{24\pi\bar{G}R_M^2\omega_1^2}\ , \tag{A.9}$$

and using the fact that $R_D^4 = R_M^4\omega_1^6 v_1^{-2}$ [cf. (4.1)]

$$\rho R_D^4 = \frac{R_M^2 \omega_1^4}{24\pi\bar{G}v_1^2} \quad . \tag{A.10}$$

At freeze-out $\rho R_D^4 = 1$, so [cf. (4.2)]

$$\frac{\omega_{1*}^4}{v_{1*}^2} = \frac{24\pi\bar{G}}{R_M^2} \quad . \tag{A.11}$$

Using the definition of $\bar{G} = R_*^2 N_{pl}$ (for $d = 1$, the Planck constant is dimensionless; N_{pl} is the Planck number) we see

$$\frac{\omega_{1*}^4}{v_{1*}^2} = \frac{24\pi R_*^2 N_{pl}}{R_M^2}$$

$$= 24\pi N_{pl} \frac{\omega_{1*}^3}{v_{1*}} \tag{A.12}$$

where the second equality comes from $R_D = R_M \omega_1^{3/2} v_1^{-1/2}$. Equation (A.12) then gives the freeze-out value of v_{1*} and ω_{1*} in terms of the Planck number (cf. 4.4.c)

$$\frac{\omega_{1*}}{v_{1*}} = 24\pi N_{pl} \quad . \tag{A.13}$$

Since $\omega_{1*}/v_{1*} = (a_1 x_*^2)^{-1}$ if we choose a Planck number, we pick the freeze-out value of $x_1 = x_{1*}$,

$$a_1 x_{1*}^2 = (24\pi N_{pl})^{-1} \quad , \tag{A.14}$$

Now the equations are invariant under the scaling $\omega(x) \to b^2\omega(x/b)$, $v(x) \to b^2 v(x/b)$, so the ratio ω_*/v_* is invariant under $x \to x/b$. Therefore (A.14) is true for any value of b, hence any value of a, since the scaling is equivalent to a different choice of a. From (A.14) we see that stopping at different x_* in the evolution and demanding freeze-out is equivalent to choosing different values of a. So we may examine the entire family of solutions by solving the system for one value of a and demanding freeze-out at different values of x_*. However it should be obvious from (4.4.c) that for definite values of d and D, there is only one value of x_* (i.e. only one among the family of solutions) that gives a particular value of R_*/R_{pl}.

The entropy, S_1, in the non-compact dimension at compactification is [cf. (5.4)]

$$S_1 = 3\omega_*^{-3} v_*^{3/2} \int_0^{x_*} \frac{\omega^2}{v} dx$$

$$= \frac{3a^2x_*^4 - a^3x_*^6}{(ax_*^2 - a^2x_*^4)^{3/2}} \tag{A.15}$$

Notice that at the second singularity, $x = 1/\sqrt{a}$, the entropy diverges.

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15. It is difficult to imagine $R_D T > 1$ at stabilization. If stabilization occurs dynamically the stress-energy contributions to the equations of motion would be bigger than the curvature terms and as the universe cools the equilibrium conditions would change. See, however, R. B. Abbott, S. M. Barr, S. D. Ellis, "Kaluza-Klein Cosmologies and Inflation", University of Washington Report 40048-03 P4 (1984).

16. This result that all low mass particles are gauge singlets happens in the 5-dimensonal Kaluza-Klein model, where the charge $e_k = kR_{pl}/R_*$ is proportional to the mass $m_k = k/R_*$ (k is an integer).

17. This solution is a Bianchi III (sometimes called $VI_h, h = -1$) solution.

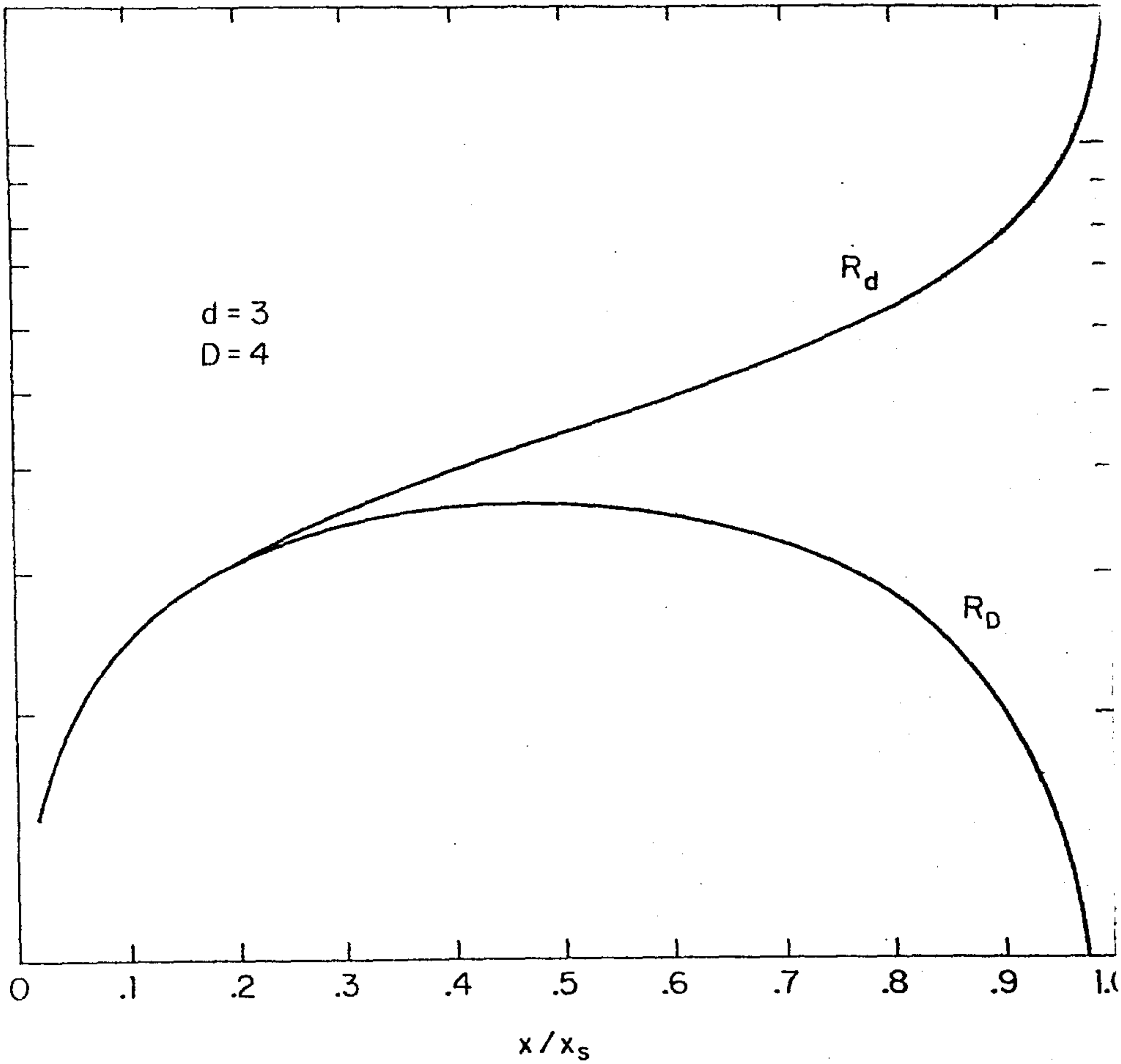

Figure 1

The compact (R_D) and open (R_d) scale factors as a function of x/x_s where x_s is the postion of the final singularity. Note that both go to zero in the same way at $x = 0$. At $x = x_s$, $R_D \to 0$ while $R_d \to \infty$.

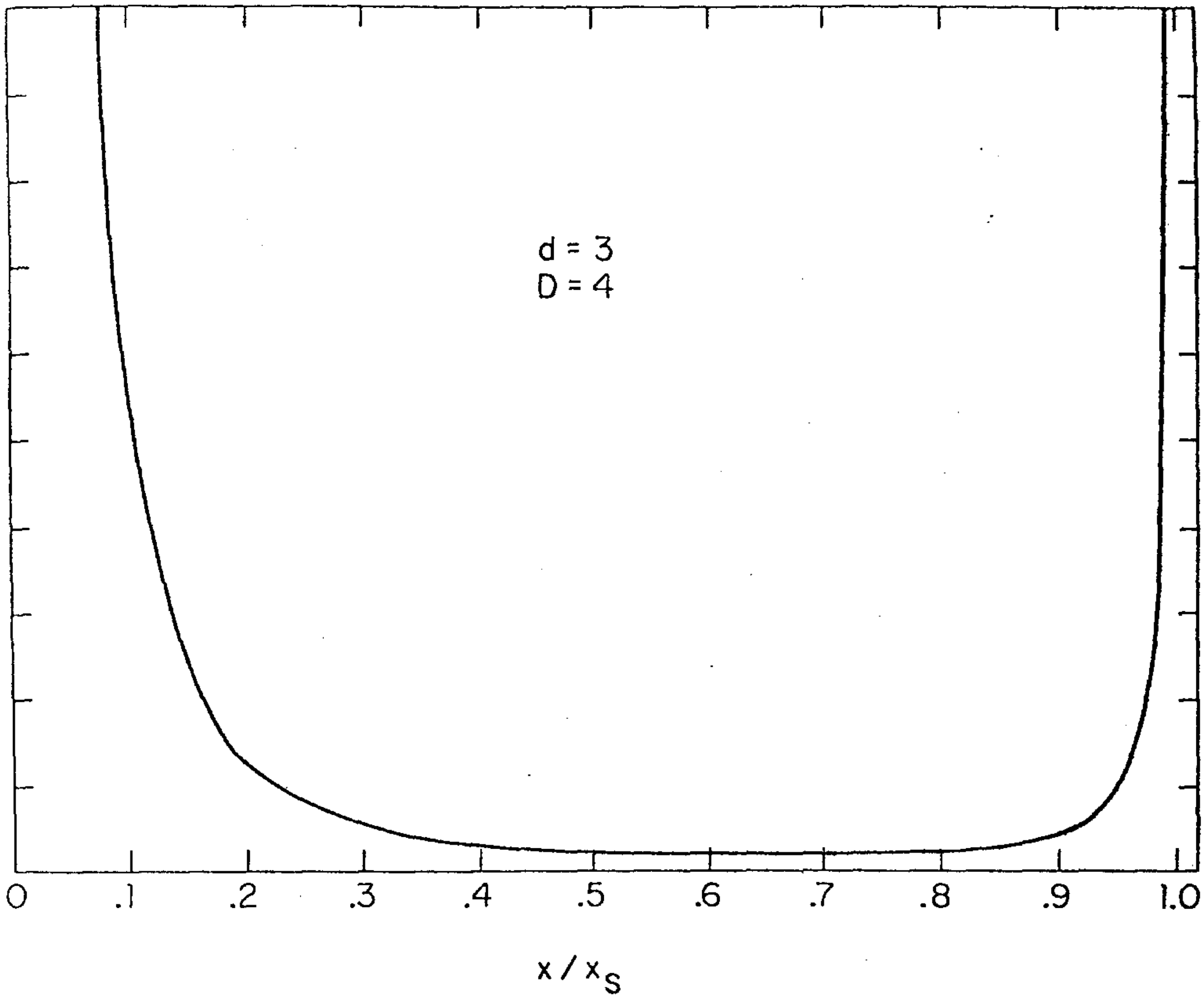

Figure 2

The inverse of the mean volume, σ^{-1}, as a function of x/x_s for the model of Figure 1. The N-dimensional entropy density, $s_7 \propto T^7$, is proportional to σ^{-1}.

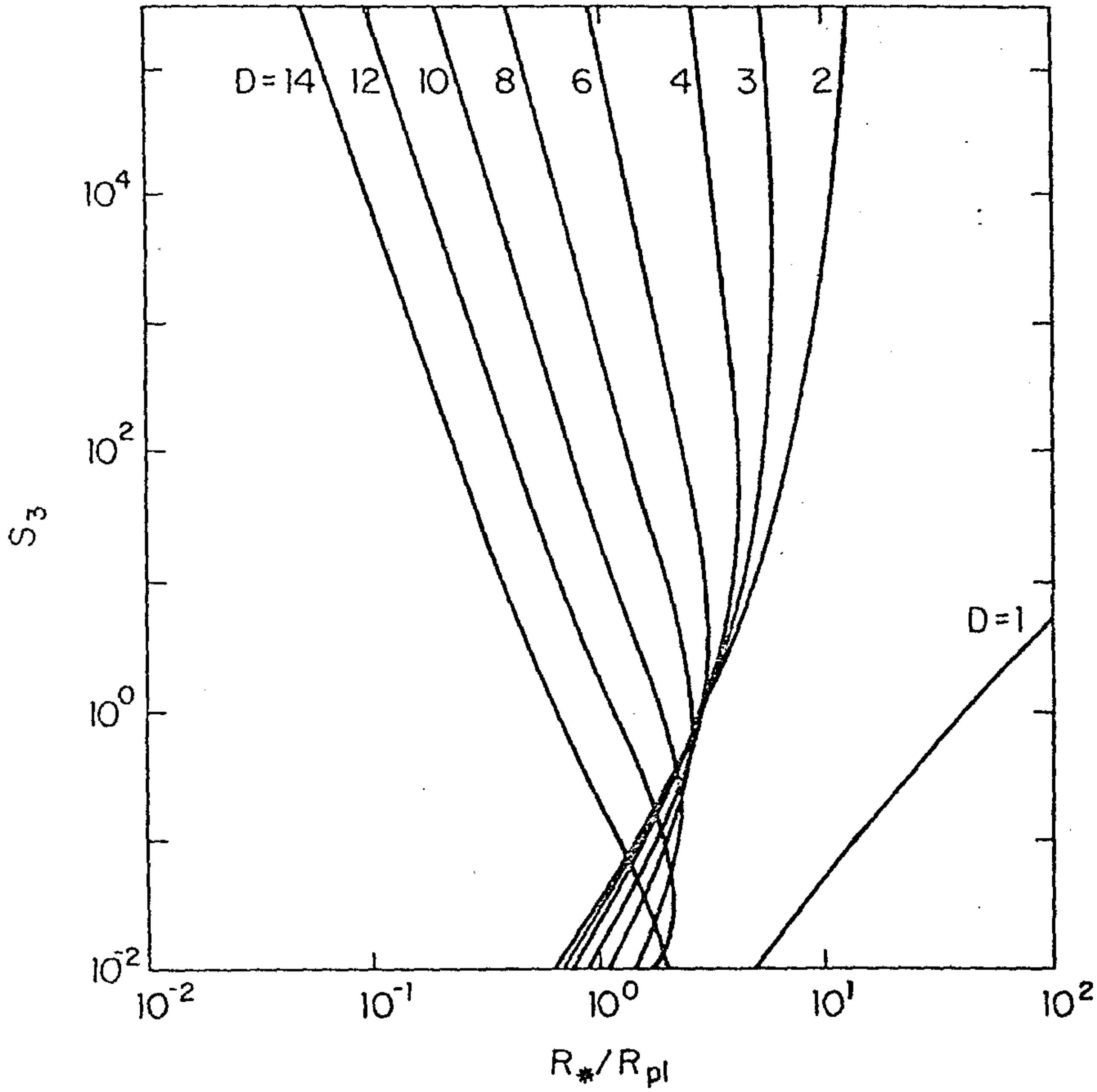

Figure 3

The entropy in the horizon volume of the 3 non-compact dimensions, S_3, as a function of R_*/R_{pl}. For $D \geq 3$, R_{kk}/R_{pl} reaches a maximum value and decreases [in accord with (4.7.b)]. Therefore R_*/R_{pl} must be small for a large S_3. For D = 1,2, S_3 increases with increasing R_*/R_{pl}.

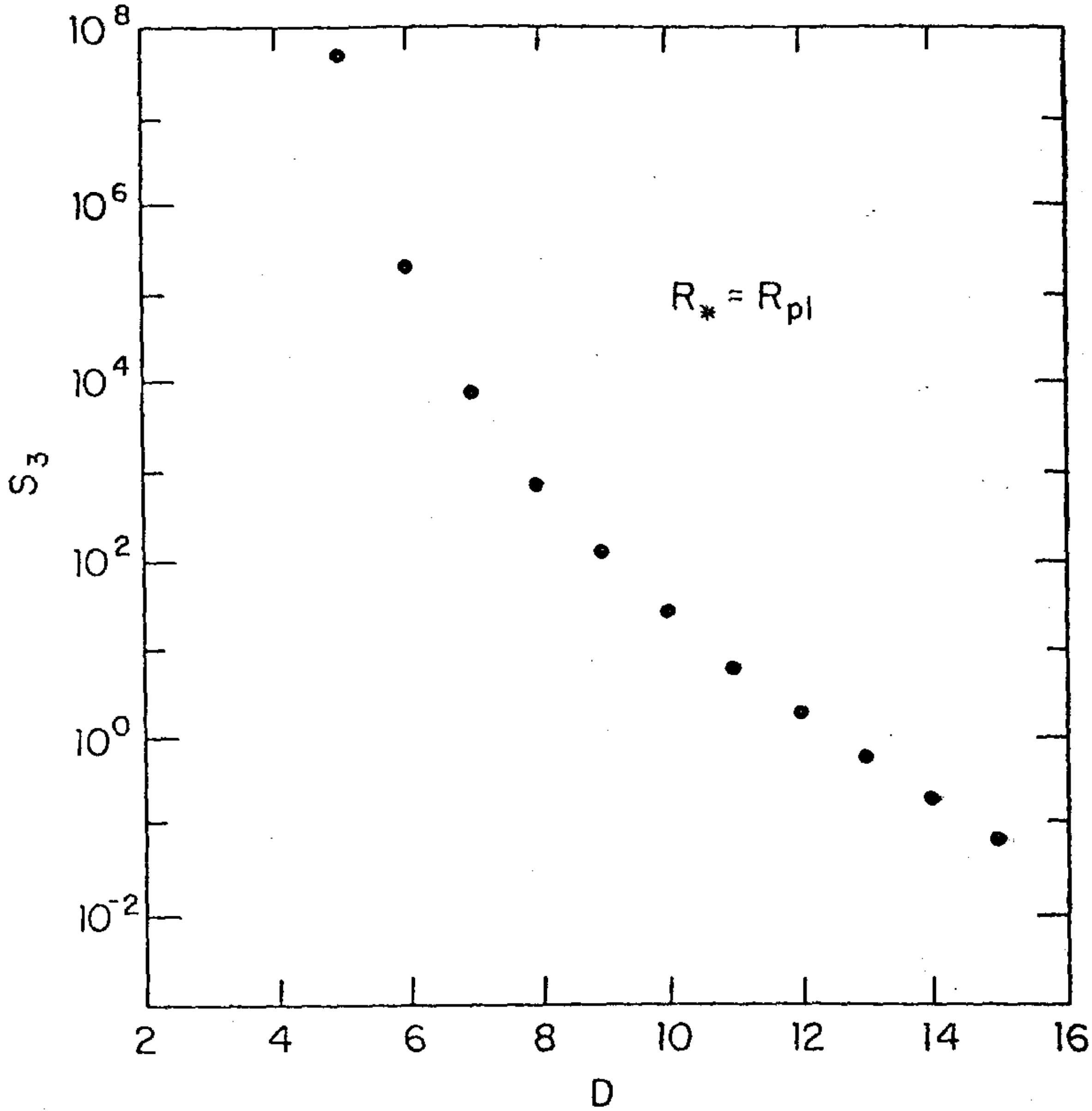

Figure 4

The entropy, S_3, as a function of the number of dimensions, D, assuming the freeze-out of the extra dimensions happens at $R_* = R_{pl}$.